

Energy evolution in time-dependent harmonic oscillator

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Abstract. The theory of adiabatic invariants has a long history, and very important implications and applications in many different branches of physics, classically and quantally, but is rarely founded on rigorous results. Here we treat the general time-dependent one-dimensional harmonic oscillator, whose Newton equation $\ddot{q} + \omega^2(t)q = 0$ cannot be solved in general. We follow the time-evolution of an initial ensemble of phase points with sharply defined energy E_0 at time $t = 0$ and calculate rigorously the distribution of energy E_1 after time $t = T$, which is fully (all moments, including the variance μ^2) determined by the first moment \bar{E}_1 . For example, $\mu^2 = E_0^2[(\bar{E}_1/E_0)^2 - (\omega(T)/\omega(0))^2]/2$, and all higher even moments are powers of μ^2 , whilst the odd ones vanish identically. This distribution function does not depend on any further details of the function $\omega(t)$ and is in this sense universal. In ideal adiabaticity $\bar{E}_1 = \omega(T)E_0/\omega(0)$, and the variance μ^2 is zero, whilst for finite T we calculate \bar{E}_1 , and μ^2 for the general case using exact WKB-theory to all orders. We prove that if $\omega(t)$ is of class C^m (all derivatives up to and including the order m are continuous) $\mu \propto T^{-(m+1)}$, whilst for class C^∞ it is known to be exponential $\mu \propto \exp(-\alpha T)$.

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1. Introduction

In time-independent (autonomous) Hamiltonian systems the total energy of the system is conserved by construction, i.e. due to the Hamilton equations of motion, and the Liouville theorem applies, because the phase flow velocity vector field has vanishing divergence. In time-dependent (nonautonomous) Hamilton systems the total energy is not conserved, whilst the Liouville theorem still applies, the phase space volume is preserved under the phase flow. If (some parameter of) the Hamilton function varies in time, the energy of the system generally also changes. But, if the changing of the parameter is very slow, on the typical time scale T , there might be a quantity I , a function of the said parameter, of the energy E and of other dynamical quantities,

which is approximately conserved. It might be even exactly conserved if $T \rightarrow \infty$, i.e. if the variation is infinitely slow, to which case we refer as the ideal adiabatic variation. Such a conserved quantity is called *adiabatic invariant*, and it plays an important role in the dynamical analysis of a long-time evolution of nonautonomous Hamilton systems.

The theory of adiabatic invariants is aimed at finding the adiabatic invariants I and analyzing the error of its preservation at finite T . Namely, the statement of exactness of I is asymptotic in the sense that the conservation is exact in the limit $T \rightarrow \infty$, whilst for finite T we see the deviation $\Delta I = I_f - I_i$ of final value of I_f from its initial value I_i and would like to calculate ΔI . Thus for finite T the final values of I will have some distribution with nonvanishing variance.

Here we just mention, as an example, that for one-dimensional harmonic oscillator it is known since Lorentz and Einstein [1] that the adiabatic invariant for $T = \infty$ is $I = E/\omega$, which is the ratio of the total energy $E = E(t)$ and the frequency of the oscillator $\omega(t)$, both being a function of time. Of course, $2\pi I$ is exactly the area in the phase plane (q, p) enclosed by the energy contour of constant E . A general introductory account of the theory of adiabatic invariants can be found in [2] and references therein, especially [3], [4]. Another quite extensive excellent review is by Henrard [5], which includes rather complete list of relevant references to the original papers.

However, in the literature this I and ΔI are not even precisely defined. It turns out that I must be generally considered as a function of the initial conditions, and then it turns out that it is conserved for some initial conditions but not for some others. As a consequence of that there is a considerable confusion about its meaning. Let us just mention the case of periodic parametric resonance (see section 5), with otherwise arbitrarily slowly changing $\omega(t)$, in one-dimensional harmonic oscillator, where the total energy of the system can grow indefinitely for certain (almost all) initial conditions, and since $\omega(t)$ is bounded, $I = E(t)/\omega(t)$ simply cannot be conserved for the said initial conditions for all times, but only for sufficiently small times. In this work we consider I as function of the initial conditions, and give a precise meaning to these and similar statements.

Therefore to be on rigorous side we must carefully define what we mean by I and ΔI . This can be done by considering an ensemble of initial conditions at time $t = 0$ just before the adiabatic process starts. Of course, there is a vast freedom in choosing such ensembles. Let us consider the one degree of freedom systems, which is the topic of this paper. If the initial conditions are on a closed contour \mathcal{K}_0 in the phase space at time $t = 0$, then at the end of the adiabatic process at time $t = T$ they are also on a closed contour \mathcal{K}_T which in general is different from \mathcal{K}_0 , but due to the Liouville theorem the area inside the contour is constant for all T . If \mathcal{K}_0 is a contour of constant energy at time $t = 0$, then \mathcal{K}_T generally is not a contour of constant energy of the Hamiltonian at time T . The final energies of the system, depending on the initial conditions, are spread and thus distributed between some minimal and maximal energy, E_{min} and E_{max} , respectively. In the said case of periodic $\omega(t)$ and parameter resonance in a harmonic oscillator the contour \mathcal{K}_T is squeezed in one direction and expanded in the other one

(the transformation is a linear map), and this contraction and expansion is exponential in time. Therefore, $I = E(T)/\omega(T)$ can not be conserved; see section 5.

Thus we must always study I as a function of initial conditions, by looking at the ensembles of initial conditions. There is a vast freedom in choosing such ensembles, but usually we do not know very much about the system except e.g. just the energy.

Therefore in an integrable conservative Hamiltonian system the most natural and the most important choice is taking as the initial ensemble all phase points uniformly distributed on the initial N -torus, i.e. uniform w.r.t. the angle variables. We call it *uniform canonical ensemble of initial conditions*. Such an ensemble has a sharply defined initial energy E_0 . Then we let the system evolve in time, not necessarily slowly, and calculate the probability distribution $P(E_1)$ of the final energy E_1 , or of other dynamical quantities. Typically E_1 is distributed on an interval (E_{min}, E_{max}) , and $P(E_1)$ is universal there, as it does not depend on any further properties of $\omega(t)$ except for \bar{E}_1 . More general ensembles of initial energies $w(E_0)$ can be described in terms of the uniform canonical ensembles, as explained in section 6.

To describe $P(E_1)$ is in general a difficult problem, but in this work we confine ourselves to the one-dimensional general time-dependent harmonic oscillator, so $N = 1$, described by the Newton equation

$$\ddot{q} + \omega^2(t)q = 0 \tag{1}$$

and work out rigorously $P(E_1)$. Given the general dependence of the oscillator's frequency $\omega(t)$ on time t the calculation of $q(t)$ is already a very difficult, in fact unsolvable, problem. In the sense of mathematical physics (1) is exactly equivalent to the one-dimensional stationary Schrödinger equation: the coordinate q appears instead of the probability amplitude ψ , time t appears instead of the coordinate x and $\omega^2(t)$ plays the role of $E - V(x) = \text{energy} - \text{potential}$. If E is greater than any local maximum of $V(x)$ then the scattering problem is equivalent to our 1D harmonic oscillator problem. In this paper we solve the above stated problem for the general one-dimensional harmonic oscillator which is a follow up paper of our Letter [6].

In performing our analysis, we shall answer the questions as to when is $I = E(T)/\omega(T)$ conserved, and if it is not conserved, what is the spread or variance μ^2 of the energy, and the higher moments etc. Then (the not sharply defined) ΔI in the literature is proportional to μ . After performing the exact analysis, we provide a powerful technique based on the WKB method [7] to calculate μ^2 , and show that it gives exact leading asymptotic terms when $T \rightarrow \infty$, and moreover, generally we can do the expansion to all orders, exactly. We treat several exactly solvable cases, and compare them with the WKB results, and finally prove the theorem as for how μ^2 behaves when $\omega(t)$ is of class \mathcal{C}^m , which means having m continuous derivatives.

We give a brief historical review of contributions to this field. After Einstein [1], Kulsrud [8] was the first to show, using a WKB-type method, that for a finite T , I is preserved to all orders, for harmonic oscillator, if all derivatives of ω vanish at the beginning and at the end of the time interval, whilst if there is a discontinuity in one of

the derivatives he estimated ΔI but did not give our explicit general expressions (37) and (38). Hertweck and Schlüter [9] did the same thing independently for a charged particle in slowly varying magnetic field for infinite time domain. Kruskal, as reported in [11], and Lenard [10] studied more general systems, whilst Gardner [11] used the classical Hamiltonian perturbation theory. Courant and Snyder [12] have studied the stability of synchrotron and analyzed I employing the transition matrix. The interest then shifted to the infinite time domain. Littlewood [13] showed for the harmonic oscillator that if $\omega(t)$ is an analytic function, I is preserved to all orders of the *adiabatic parameter* $\epsilon = 1/T$. Kruskal [14] developed the asymptotic theory of Hamiltonian and other systems with all solutions nearly periodic. Lewis [15], using the Kruskal's method, discovered a connection between I of the 1-dim harmonic oscillator and another nonlinear differential equation. Later on Symon [16] used the Lewis's results to calculate the (canonical) ensemble average of the I and its variance, which is the analogue of our \bar{E}_1 and μ^2 . Finally, Knorr and Pfirsch [17] proved $\Delta I \propto \exp(-\text{const}/\epsilon)$. Meyer [18], [19] relaxed some conditions and calculated the constant *const*. Exponential preservation of I for an analytic ω on $(-\infty, +\infty)$ with constant limits at $t \rightarrow \pm\infty$, is thus well established [3].

2. Transition map and general exact considerations

We begin by defining the system by giving its Hamilton function $H = H(q, p, t)$, whose numerical value $E(t)$ at time t is precisely the total energy of the system at time t , and for the general time-dependent one-dimensional harmonic oscillator this is

$$H = \frac{p^2}{2M} + \frac{1}{2}M\omega^2(t)q^2, \quad (2)$$

where q, p, M, ω are the coordinate, the momentum, the mass and the frequency of the linear oscillator, respectively. The dynamics is linear in q, p , as described by (1), but nonlinear as a function of $\omega(t)$ and therefore is subject to the nonlinear dynamical analysis. By using the index 0 and 1 we denote the initial ($t = t_0$) and final ($t = t_1$) value of the variables, and by $T = t_1 - t_0$ we denote the length of the time interval of changing the parameters of the system.

We consider the phase flow map (we shall call it transition map)

$$\Phi : \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \mapsto \begin{pmatrix} q_1 \\ p_1 \end{pmatrix}. \quad (3)$$

Because equations of motion are linear in q and p , and since the system is Hamiltonian, Φ is a linear area preserving map, that is,

$$\Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (4)$$

with $\det(\Phi) = ad - bc = 1$. Let $E_0 = H(q_0, p_0, t = t_0)$ be the initial energy and $E_1 = H(q_1, p_1, t = t_1)$ be the final energy, that is,

$$E_1 = \frac{1}{2} \left(\frac{(cq_0 + dp_0)^2}{M} + M\omega_1^2(aq_0 + bp_0)^2 \right). \quad (5)$$

Introducing the new coordinates, namely the action $I = E/\omega$ and the angle ϕ ,

$$q_0 = \sqrt{\frac{2E_0}{M\omega_0^2}} \cos \phi, \quad p_0 = \sqrt{2ME_0} \sin \phi \quad (6)$$

from (5) we obtain

$$E_1 = E_0(\alpha \cos^2 \phi + \beta \sin^2 \phi + \gamma \sin 2\phi), \quad (7)$$

where

$$\alpha = \frac{c^2}{M^2\omega_0^2} + a^2\frac{\omega_1^2}{\omega_0^2}, \quad \beta = d^2 + \omega_1^2 M^2 b^2, \quad \gamma = \frac{cd}{M\omega_0} + abM\frac{\omega_1^2}{\omega_0}. \quad (8)$$

Given the uniform probability distribution of initial angles ϕ equal to $1/(2\pi)$, which defines our initial *uniform canonical ensemble* at time $t = t_0$, we can now calculate the averages. Thus

$$\bar{E}_1 = \frac{1}{2\pi} \oint E_1 d\phi = \frac{E_0}{2}(\alpha + \beta). \quad (9)$$

That yields $E_1 - \bar{E}_1 = E_0(\delta \cos 2\phi + \gamma \sin 2\phi)$ and

$$\mu^2 = \overline{(E_1 - \bar{E}_1)^2} = \frac{E_0^2}{2} (\delta^2 + \gamma^2), \quad (10)$$

where we have denoted $\delta = (\alpha - \beta)/2$. It follows from (8), (9) that we can write (10) also in the form

$$\mu^2 = \overline{(E_1 - \bar{E}_1)^2} = \frac{E_0^2}{2} \left[\left(\frac{\bar{E}_1}{E_0} \right)^2 - \left(\frac{\omega_1}{\omega_0} \right)^2 \right]. \quad (11)$$

As we shall see (see subsection 4.2), in an ideal adiabatic process $\mu = 0$, and therefore $E_1 = \bar{E}_1 = \omega_1 E_0 / \omega_0$, and consequently $P(E_1)$ is a delta function,

$$P(E_1) = \delta(E_1 - \omega_1 E_0 / \omega_0). \quad (12)$$

Now we calculate higher moments of $P(E_1)$. Let us show that in general for arbitrary positive integer m

$$\overline{(E_1 - \bar{E}_1)^{2m-1}} = 0 \quad (13)$$

and

$$\overline{(E_1 - \bar{E}_1)^{2m}} = \frac{(2m-1)!!}{m!} \left(\overline{(E_1 - \bar{E}_1)^2} \right)^m. \quad (14)$$

It is easy to check the correctness of (13), so we prove only (14). Indeed,

$$\begin{aligned} \overline{(E_1 - \bar{E}_1)^{2m}} &= E_0^{2m} \overline{(\gamma \sin 2\phi + \delta \cos 2\phi)^{2m}} = \\ &E_0^{2m} \sum_{k=0}^{2m} \binom{2m}{k} \overline{(\gamma^{2m-k} \delta^k \sin^{2m-k} 2\phi \cos^k 2\phi)}. \end{aligned} \quad (15)$$

Note that for odd $k = 2l - 1$

$$\overline{\sin^{2m-k} 2\phi \cos^k 2\phi} = 0. \quad (16)$$

Therefore we can rewrite (15) in the form

$$\overline{(E_1 - \bar{E}_1)^{2m}} = E_0^{2m} \sum_{l=0}^m \binom{2m}{2l} \overline{(\gamma^{2m-2l} \delta^{2l} \sin^{2m-2l} 2\phi \cos^{2l} 2\phi)}. \quad (17)$$

Using formula 2.512 of [20] we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \sin^{2(m-l)} 2\phi \cos^{2l} 2\phi d\phi = \frac{(2l-1)!!(2m-2l-1)!!}{2^m m!}. \quad (18)$$

The substitution of this value into (17) yields

$$\overline{(E_1 - \bar{E}_1)^{2m}} = E_0^{2m} \sum_{l=0}^m \binom{2m}{2l} \left(\gamma^{2m-2l} \delta^{2l} \frac{(2l-1)!!(2m-2l-1)!!}{2^m m!} \right) \quad (19)$$

$$= E_0^{2m} \sum_{l=0}^m \left(\gamma^{2m-2l} \delta^{2l} \frac{(2m)!}{2^m m! (2l)!! (2m-2l)!!} \right) \quad (20)$$

$$= E_0^{2m} \frac{(2m-1)!!}{2^m m!} \sum_{l=0}^m \binom{m}{l} \gamma^{2m-2l} \delta^{2l} \quad (21)$$

$$= \frac{(2m-1)!!}{2^m m!} (\gamma^2 + \delta^2)^m E_0^{2m} = \frac{(2m-1)!!}{m!} \left(\overline{(E_1 - \bar{E}_1)^2} \right)^m, \quad (22)$$

that is, the formula (14) holds. Thus $2m$ -th moment of $P(E_1)$ is equal to $(2m-1)!! \mu^{2m}/m!$, and therefore, indeed, all moments of $P(E_1)$ are uniquely determined by the first moment \bar{E}_1 .

We now mention that the final energy distribution function written down as

$$P(E_1) = \frac{1}{2\pi} \sum_{j=1}^4 \left| \frac{d\phi}{dE_1} \right|_{\phi=\phi_j(E_1)} \quad (23)$$

cannot be calculated analytically in a closed form in any useful way, because it boils down to finding the roots of a quartic polynomial, so we do not try to do that here, although numerically it shows interesting aspects (see figure 3). Obviously, $P(E_1)$ is in this sense universal, because it depends only on the average final energy \bar{E}_1 and the ratio ω_1/ω_0 of the final and initial frequencies, and does not depend otherwise on any details of $\omega(t)$. It has a finite interval as its support, between the lower limit E_{min} and the upper limit E_{max} , it is an even function there w.r.t. the mean value $\bar{E}_1 = (E_{min} + E_{max})/2$,

and from $E_1 - \bar{E}_1 = E_0(\delta \cos 2\phi + \gamma \sin 2\phi)$ it is easy to show that $E_{min} = \bar{E}_1 - \mu\sqrt{2}$ and $E_{max} = \bar{E}_1 + \mu\sqrt{2}$, so that $E_{max} - E_{min} = \mu\sqrt{8} = 2E_0\sqrt{\gamma^2 + \delta^2}$. At both extreme values it has an integrable singularity of the type $1/\sqrt{x}$. This singularity stems from a projection of the final ensemble at $t_1 = T$ onto the curves of constant final energies E_1 of $H(q, p, t_1)$. In between for every value of $E_1 = \text{const} = E_1(\phi)$, this equation has four solutions, namely $\phi_1, \phi_2, \phi_3, \phi_4$, and thus we have to sum up all four contributions in the general formula (23). Of course, all that we say here for the distribution of energies E_1 holds true also for the final action, the adiabatic invariant $I_1 = E_1/\omega_1$. It is perhaps worthwhile to mention that the moments of our distribution according to (14) grow as $2^m/\sqrt{\pi m}$, whilst e.g. in the Gaussian distribution they grow much faster, namely as $2^m\Gamma(m + 1/2)/\sqrt{\pi}$, where $\Gamma(x)$ denotes the gamma function.

Expression (11) is positive definite by definition and this leads to the first interesting conclusion: In full generality (no restrictions on the function $\omega(t)$!) we have always $\bar{E}_1 \geq E_0\omega_1/\omega_0$ and therefore the final value of the adiabatic invariant (for the average energy!) $\bar{I}_1 = \bar{E}_1/\omega_1$ is always greater or equal to the initial value $I_0 = E_0/\omega_0$. In other words, the value of the adiabatic invariant at the mean value of the energy never decreases, which is a kind of irreversibility statement. Moreover, it is conserved only for infinitely slow processes $T = \infty$, which is an ideal adiabatic process, for which $\mu = 0$. See section 4.3. For periodic processes $\omega_1 = \omega_0$ we see that always $\bar{E}_1 \geq E_0$, so the mean energy never decreases. See section 5. The other extreme opposite to $T = \infty$ is the instantaneous ($T = 0$) jump where ω_0 switches to ω_1 discontinuously, whilst q and p remain continuous, and this results in $a = d = 1$ and $b = c = 0$, and then we find

$$\bar{E}_1 = \frac{E_0}{2}(\frac{\omega_1^2}{\omega_0^2} + 1), \quad \mu^2 = \frac{E_0^2}{8} \left[\frac{\omega_1^2}{\omega_0^2} - 1 \right]^2. \quad (24)$$

Below we shall treat the special case with $\omega_1^2 = 2\omega_0^2$, and thus will find $\mu^2/E_0^2 = 1/8 = 0.125$.

Our general study now focuses on the calculation of the transition map (4), namely its matrix elements a, b, c, d . Starting from the Hamilton function (2) and its Newton equation (1) we consider two linearly independent solutions $\psi_1(t)$ and $\psi_2(t)$ and introduce the matrix

$$\Psi(t) = \begin{pmatrix} \psi_1(t) & \psi_2(t) \\ M\dot{\psi}_1(t) & M\dot{\psi}_2(t) \end{pmatrix}. \quad (25)$$

Consider a solution $\hat{q}(t)$ of (1) such that

$$\hat{q}(t_0) = q_0, \quad \dot{\hat{q}}(t_0) = p_0/M. \quad (26)$$

Because ψ_1 and ψ_2 are linearly independent, we can look for $\hat{q}(t)$ in the form

$$\hat{q}(t) = A\psi_1(t) + B\psi_2(t). \quad (27)$$

Then A and B are determined by

$$\begin{pmatrix} A \\ B \end{pmatrix} = \Psi^{-1}(t_0) \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}. \quad (28)$$

Let $q_1 = \hat{q}(t_1)$, $p_1 = M\dot{\hat{q}}(t_1)$. Then from (26)–(28) we see that

$$\begin{pmatrix} q_1 \\ p_1 \end{pmatrix} = \Psi(t_1)\Psi^{-1}(t_0) \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}. \quad (29)$$

We recognize the matrix on the right-hand side of (29) as the transition map Φ , that is,

$$\Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Psi(t_1)\Psi^{-1}(t_0). \quad (30)$$

3. Some exactly solvable special cases

The theory of adiabatic invariants is rarely founded on rigorous results, therefore the study of exactly solvable cases is of fundamental importance, namely we can use them to test various analytic approximations and also the accuracy of numerical calculations. Because the ideal adiabatic processes are infinitely slow and refer to the limit $T \rightarrow \infty$ we must deal necessarily with the asymptotic behaviour of dynamical systems, which is difficult to approximate, because all simple minded perturbational and other approximation techniques fail (break down) after some finite time, and so typically give wrong predictions for the asymptotic behaviour in the limit $T \rightarrow \infty$. The same difficulty occurs in numerical calculations. We shall see that the WKB methods [7] can be successfully applied. In this section we deal with three different models for the function $\omega^2(t)$.

3.1. The linear model: class \mathcal{C}^0

We assume that function $\omega^2(t)$ is a piecewise linear function of the form

$$\omega^2(t) = \begin{cases} \omega_0^2 & \text{if } t \leq 0 \\ \omega_0^2 + \frac{(\omega_1^2 - \omega_0^2)}{T} t & \text{if } 0 < t < T \\ \omega_1^2 & \text{if } t \geq T \end{cases}. \quad (31)$$

Thus $\omega(t)$ has discontinuous first derivative at $t = 0$ and $t = T$, and belongs to the class \mathcal{C}^0 . Introducing the notation $\tilde{a} = \omega_0^2$, $\tilde{b} = \omega_1^2 - \omega_0^2$ we obtain that on the interval $(0, T)$ the equation (1) has the form

$$\ddot{q} + \left(\tilde{a} + \frac{\tilde{b}t}{T}\right)q = 0. \quad (32)$$

Two linear independent solutions of (32) are given by the Airy functions:

$$\psi_1(t) = Ai\left(\frac{\tilde{b}t + \tilde{a}T}{\tilde{b}^{2/3}T^{1/3}}\right) \quad (33)$$

and

$$\psi_2(t) = Bi\left(\frac{\tilde{b}t + \tilde{a}T}{\tilde{b}^{2/3}T^{1/3}}\right). \quad (34)$$

The elements a, b, c, d of the matrix Φ_T are defined by the equation (29) and (30).

The exact analytic expression for

$$\mu^2 = \overline{(E_1 - \bar{E}_1)^2} = \frac{E_0}{2} (\delta^2 + \gamma^2) \quad (35)$$

is very complex, and we do not show it here. However, for $\omega_0^2 = 1, \omega_1^2 = 2, E_0 = 1$, using the asymptotic expansion 10.4.60,62,64,66 of [21] (pp.448-449), we obtain the following approximation

$$\overline{(E_1 - \bar{E}_1)^2} \approx \frac{\epsilon^2}{128} \left(9 - 4\sqrt{2} \cos\left(\frac{4 - 8\sqrt{2}}{3\epsilon}\right) \right), \quad (36)$$

where we introduce **the adiabatic parameter** ϵ ,

$$\epsilon = \frac{1}{T}. \quad (37)$$

We see in figure 1 that the exact expression (35) and its leading asymptotic approximation (36) practically coincide, which demonstrates the power of the asymptotic expansion of the relevant expressions containing the Airy functions. Observe that the decay of μ^2 to zero as $\epsilon \rightarrow 0$ is oscillatory but quadratic on the average, namely as $y = \frac{9}{128}\epsilon^2$, which means that μ goes to zero linearly with the adiabatic parameter ϵ . This is always the case when $\omega(t)$ is of class \mathcal{C}^0 . As we will see, in general, if $\omega(t)$ is of class \mathcal{C}^m , then μ goes to zero oscillatory but in the mean as a power ϵ^{m+1} . This theorem will be proven in subsection 4.2 using the exact formulation of the WKB method, applied to the relevant (but arbitrarily high) order. We shall see in subsection 4.2, equation (68,) that the leading WKB term precisely reproduces the exact leading term in (36). In figure 2 we show the same thing as in figure 1 but on the larger scale of ϵ .

We now compute the final energy distribution function

$$P(E_1) = \frac{1}{2\pi} \sum_{i=1}^4 \left| \frac{d\phi}{dE_1} \right|_{\phi=\phi_j(E_1)}.$$

Note that

$$\frac{d\phi}{dE_1} = \frac{1}{E'_1(\phi)}. \quad (38)$$

Using the substitution

$$u = \tan(\phi/2) \quad (39)$$

we obtain from (7)

$$(1 + u^2)^2 E_1 = E_0 (\alpha(1 - u^2)^2 + 4\beta u^2 + 4\gamma u(1 - u^2)). \quad (40)$$

Resolving this equation with respect to u yields four branches of the function $\phi(E_1)$. In figure 3 we show the emerging distribution function as an illustration.

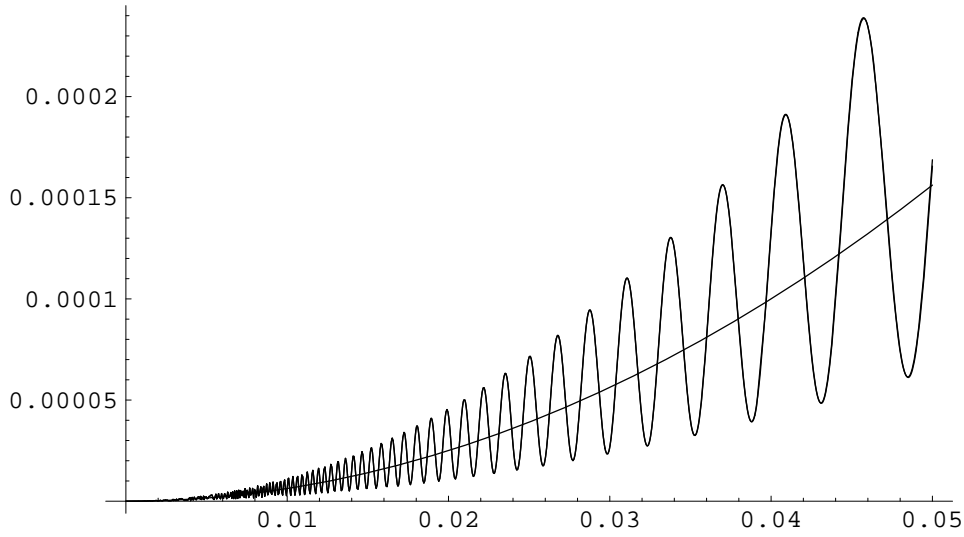


Figure 1. $\overline{(E_1 - \bar{E}_1)^2}$ for $0 < \epsilon < 0.05$; the lines of the exact expression (35) and the asymptotics (36) practically coincide; the non-oscillating thin line is the parabola $y = \frac{9}{128}\epsilon^2$.

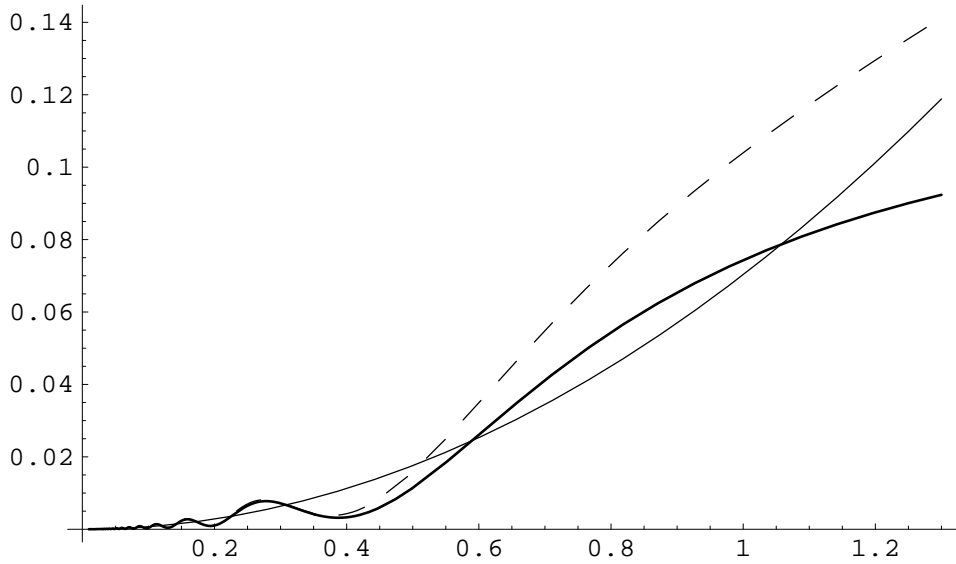


Figure 2. $\overline{(E_1 - \bar{E}_1)^2}$ for $0 < \epsilon < 1.2$; the lines of the exact expression (35) (full) and of the asymptotics (36) (dashed) practically coincide for $\epsilon \leq 0.3$; the non-oscillating thin line is the parabola $y = \frac{9}{128}\epsilon^2$. One can show that μ^2 goes to $1/8 = 0.125$ when $\epsilon \rightarrow \infty$, which means $T \rightarrow 0$, which means the instantaneous jump of $\omega_0 = 1$ to $\omega_1 = \sqrt{2}$.

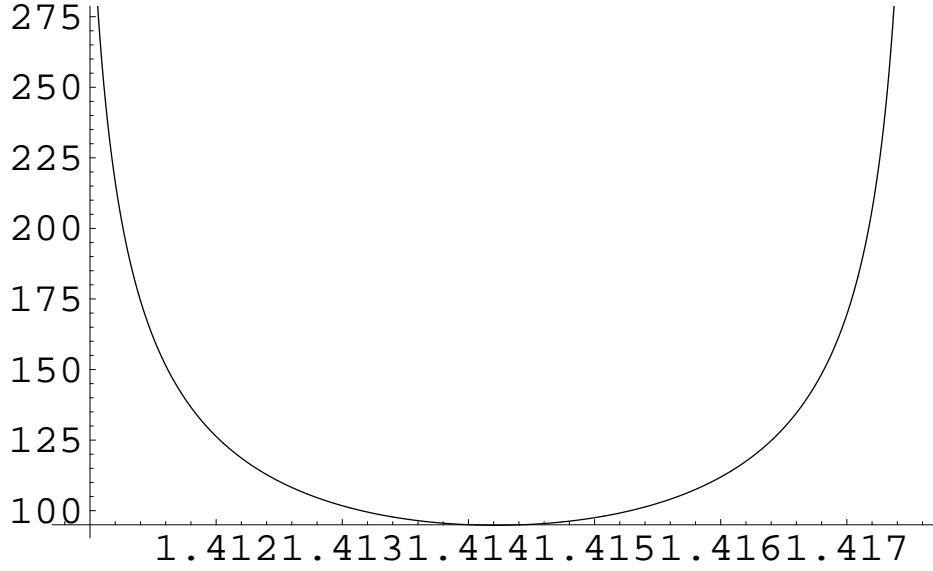


Figure 3. The distribution function $P(E_1)$ for the linear model of subsection 3.1, for $E_0 = 1, \omega_0^2 = 1, \omega_1^2 = 2, \epsilon = 0.01$.

3.2. The harmonic model: class \mathcal{C}^1

Now we consider the case when $\omega(t)$ is of class \mathcal{C}^1 , namely we use the specific model

$$\omega^2(t) = \begin{cases} \tilde{a} & \text{if } t \leq 0 \\ \tilde{b} - (\tilde{b} - \tilde{a}) \cos\left(\frac{\pi t}{T}\right) & 0 < t < T \\ 2\tilde{b} - \tilde{a} & \text{if } t \geq T \end{cases}, \quad (41)$$

where $\tilde{a} = \omega_0^2$, $2\tilde{b} - \tilde{a} = \omega_1^2$. Then the equation (1) has the form

$$\ddot{q} + \left(\tilde{b} - (\tilde{b} - \tilde{a}) \cos\left(\frac{\pi t}{T}\right) \right) q = 0 \quad (42)$$

and the fundamental solutions are represented by the Mathieu functions:

$$\psi_1(t) = ce\left(\frac{4\tilde{b}T^2}{\pi^2}, \frac{2(\tilde{b} - \tilde{a})T^2}{\pi^2}, \frac{\pi t}{2T}\right) \quad (43)$$

and

$$\psi_2(t) = se\left(\frac{4\tilde{b}T^2}{\pi^2}, \frac{2(\tilde{b} - \tilde{a})T^2}{\pi^2}, \frac{\pi t}{2T}\right). \quad (44)$$

In this model the first derivative of $\omega(t)$ vanishes at $t = 0$ and $t = T$ and is continuous there, whilst the second derivative is discontinuous, therefore $\omega(t)$ belongs to the class \mathcal{C}^1 .

Here we find that indeed μ^2 goes to zero oscillating but in the mean as ϵ^4 , as shown in figure 4. The exact analytic asymptotic behaviour of the Mathieu functions needed for our calculations is not known, unlike for the Airy functions, but we compare the exact (numerical result) with the WKB approximation expounded in section 4, see the final general formula (98) of subsection 4.3.

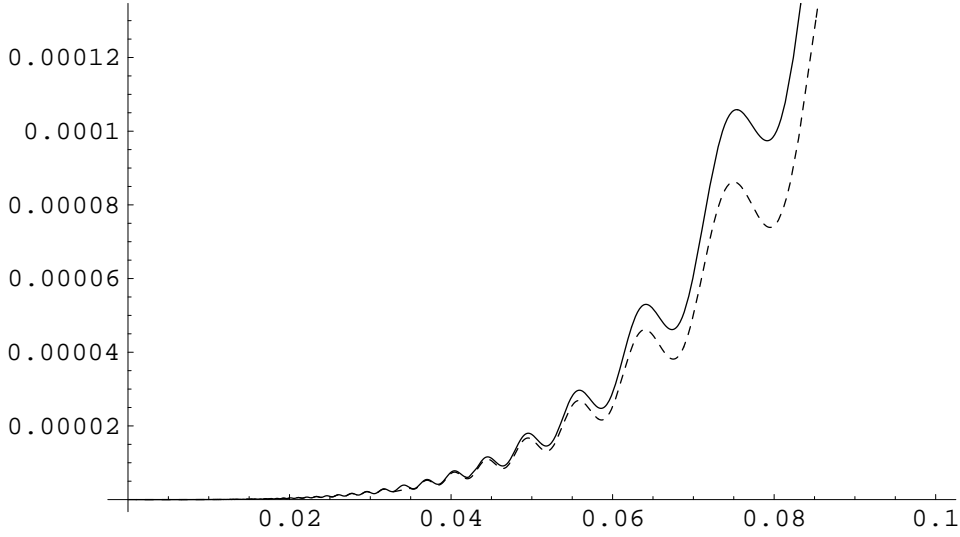


Figure 4. We show, for the harmonic model of subsection 3.2, $\overline{(E_1 - \bar{E}_1)^2}$ for $0 < \epsilon < 0.1$. The exact result is represented by the full line, whilst the dashed curve is the curve $0.056 \epsilon^4 (41 + 9 \cos(\frac{2.78}{\epsilon}))$, obtained by the WKB method, equation (98) of subsection 4.3.

3.3. The analytic model: class \mathcal{C}^∞

Consider now the case when

$$\omega^2(t) = \frac{1 + a e^{\alpha t}}{1 + e^{\alpha t}}, \quad (45)$$

that is, (1) is of the form

$$\ddot{q} + \frac{1 + a e^{\alpha t}}{1 + e^{\alpha t}} q = 0. \quad (46)$$

(Please do not confuse these a and α with the usage in the main text.) Then two fundamental solutions of (46) are represented using the hypergeometric function ${}_2F_1$ as follows:

$$\psi_1(t) = e^{-it} {}_2F_1\left(-\frac{i}{\alpha} - \frac{i\sqrt{a}}{\alpha}, -\frac{i}{\alpha} + \frac{i\sqrt{a}}{\alpha}, 1 - \frac{2i}{\alpha}, -e^{\alpha t}\right) \quad (47)$$

and

$$\psi_2(t) = e^{it} {}_2F_1\left(\frac{i}{\alpha} - \frac{i\sqrt{a}}{\alpha}, \frac{i}{\alpha} + \frac{i\sqrt{a}}{\alpha}, 1 + \frac{2i}{\alpha}, -e^{\alpha t}\right). \quad (48)$$

Using (47) and (48) we compute the matrix Φ defined by (30) with $t_0 = -\frac{10}{\alpha}$ and $t_1 = \frac{10}{\alpha}$, so $T = t_1 - t_0 = \frac{20}{\alpha}$ (note that the most essential change of the function $\omega(t)$ is on the interval $(-\frac{2}{\alpha}, \frac{2}{\alpha})$). For the case $a = 2$ the graph of $\overline{(E_1 - \bar{E}_1)^2}$ as the function of $\epsilon = 1/T$ is displayed in figure 5 as continuous line. The approximation to the graph, $y = 4.174e^{-0.634/\epsilon}$, computed using the least square method, is displayed in the figure as dashed line. We clearly see that the behaviour of μ^2 is exponential.

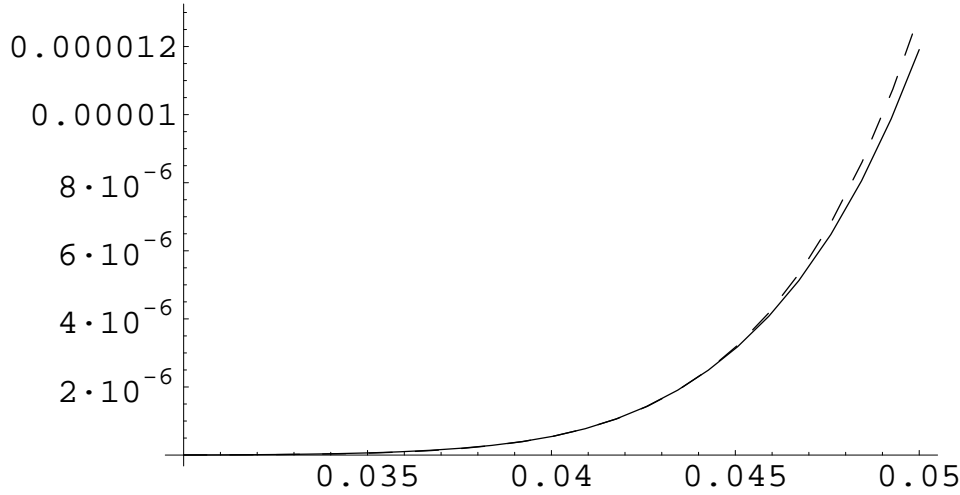


Figure 5. The variance $\overline{(E_1 - \bar{E}_1)^2}$ of the energy for the analytic model (45), for $0.03 < \epsilon < 0.05$. The dashed curve is approximation $y = 4.174e^{-0.634/\epsilon}$.

4. Application of the WKB method for the calculation of the transition map

4.1. General and exact considerations

In this section we proceed with the calculation of the transition map Φ in the general case, and because (1) is generally not solvable, we have ultimately to resort to some approximations. Since the adiabatic limit $\epsilon \rightarrow 0$ is the asymptotic regime that we would like to understand, the application of the rigorous WKB theory (up to all orders) is most convenient, and usually it turns out that the leading asymptotic terms are well described by just the leading WKB terms if ϵ is sufficiently small. In using the WKB method we refer to our work [7], where we have derived the explicit analytic expressions for all WKB orders in closed form, except for the exact rational coefficients, which can be easily obtained from a recurrence formula. ‡

We introduce re-scaled and dimensionless time λ

$$\lambda = \epsilon t, \quad \epsilon = 1/T, \quad (49)$$

so that (1) is transformed to the equation

$$\epsilon^2 q''(\lambda) + \omega^2(\lambda) q(\lambda) = 0. \quad (50)$$

By prime we denote the differentiation w.r.t. λ . Let $q_+(\lambda)$ and $q_-(\lambda)$ be two linearly

‡ There is substantial literature on WKB method, which due to limited space cannot be reviewed here. But we should mention the classic works by N. Fröman and P.O. Fröman, who have found a number of interesting relationships, e.g. a relation between the even and odd order terms [22], although we do not use it here, so that our exposition is selfcontained.

independent solutions of (50). Then the matrix (25) takes the form

$$\Psi_\lambda = \begin{pmatrix} q_+(\lambda) & q_-(\lambda) \\ \epsilon M q'_+(\lambda) & \epsilon M q'_-(\lambda) \end{pmatrix} \quad (51)$$

and taking into account that $\lambda_0 = \epsilon t_0, \lambda_1 = \epsilon t_1$ we obtain for the matrix (4) the expression

$$\Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Psi_\lambda(\lambda_1) \Psi_\lambda^{-1}(\lambda_0). \quad (52)$$

We now use the WKB method in order to obtain the coefficients a, b, c, d of the matrix Φ . To do so, we look for solution of (50) in the form

$$q(\lambda) = w \exp \left\{ \frac{1}{\epsilon} \sigma(\lambda) \right\} \quad (53)$$

where $\sigma(\lambda)$ is a complex function that satisfies the differential equation

$$(\sigma'(\lambda))^2 + \epsilon \sigma''(\lambda) = -\omega^2(\lambda) \quad (54)$$

and w is some constant with dimension of length. The WKB expansion for the phase is

$$\sigma(\lambda) = \sum_{k=0}^{\infty} \epsilon^k \sigma_k(\lambda). \quad (55)$$

Substituting (55) into (54) and comparing like powers of ϵ gives the recursion relation

$$\sigma_0'^2 = -\omega^2(\lambda), \quad \sigma_n' = -\frac{1}{2\sigma_0'} \left(\sum_{k=1}^{n-1} \sigma_k' \sigma_{n-k}' + \sigma_{n-1}'' \right). \quad (56)$$

Here we apply our WKB notation and formalism from our work [7] and we can choose $\sigma'_{0,+}(\lambda) = i\omega(\lambda)$ or $\sigma'_{0,-}(\lambda) = -i\omega(\lambda)$. That results in two linearly independent solutions of (50) given by the WKB expansions with the coefficients

$$\sigma_{0,\pm}(\lambda) = \pm i \int_{\lambda_0}^{\lambda} \omega(x) dx, \quad \sigma_{1,\pm}(\lambda) = -\frac{1}{2} \log \frac{\omega(\lambda)}{\omega(\lambda_0)}, \quad (57)$$

$$\sigma_{2,\pm} = \pm \frac{i}{8} \int_{\lambda_0}^{\lambda} \frac{3\omega'(x)^2 - 2\omega(x)\omega''(x)}{\omega(x)^3} dx, \quad \dots \quad (58)$$

Since $\omega(\lambda)$ is a real function we deduce from (56) that all functions σ'_{2k+1} are real and all functions σ'_{2k} are pure imaginary and $\sigma'_{2k,+} = -\sigma'_{2k,-}$, $\sigma'_{2k+1,+} = \sigma'_{2k+1,-}$ where $k = 0, 1, 2, \dots$, and thus we have $\sigma'_+ = A(\lambda) + iB(\lambda)$, $\sigma'_- = A(\lambda) - iB(\lambda)$ where $A(\lambda) = \sum_{k=0}^{\infty} \epsilon^{2k+1} \sigma'_{2k+1}(\lambda)$, $B(\lambda) = -i \sum_{k=0}^{\infty} \epsilon^{2k} \sigma'_{2k,+}(\lambda)$ are both real quantities. Integration of the above equations yields

$$\sigma_+ = r(\lambda) + is(\lambda), \quad \sigma_- = r(\lambda) - is(\lambda), \quad (59)$$

where $r(\lambda) = \int_{\lambda_0}^{\lambda} A(x) dx$, $s(\lambda) = \int_{\lambda_0}^{\lambda} B(x) dx$. Below we shall denote $s_1 = s(\lambda_1)$. To simplify the expressions let us denote $A_0 = A(\lambda_0)$, $A_1 = A(\lambda_1)$, $B_0 = B(\lambda_0)$ and $B_1 = B(\lambda_1)$.

Using this notation we find that the elements of the matrix Φ_λ have the following form:

$$\begin{aligned} a &= -\frac{e^{\frac{r(\lambda_1)}{\epsilon}}}{B_0} \left[A_0 \sin\left(\frac{s_1}{\epsilon}\right) - B_0 \cos\left(\frac{s_1}{\epsilon}\right) \right], \\ b &= \frac{e^{\frac{r(\lambda_1)}{\epsilon}}}{MB_0} \sin\left(\frac{s_1}{\epsilon}\right), \\ c &= -\frac{M}{B_0} e^{\frac{r(\lambda_1)}{\epsilon}} \left[(A_0 A_1 + B_0 B_1) \sin\left(\frac{s_1}{\epsilon}\right) \right. \\ &\quad \left. + (A_0 B_1 - A_1 B_0) \cos\left(\frac{s_1}{\epsilon}\right) \right], \\ d &= \frac{e^{\frac{r(\lambda_1)}{\epsilon}}}{B_0} \left[A_1 \sin\left(\frac{s_1}{\epsilon}\right) + B_1 \cos\left(\frac{s_1}{\epsilon}\right) \right]. \end{aligned} \quad (60)$$

Therefore for the quantities α and β defined by (8) we obtain

$$\begin{aligned} \alpha &= \frac{e^{2\frac{r(\lambda_1)}{\epsilon}}}{\omega_0^2 B_0^2} \left[\omega_1^2 \left(A_0 \sin\left(\frac{s_1}{\epsilon}\right) - B_0 \cos\left(\frac{s_1}{\epsilon}\right) \right)^2 + \right. \\ &\quad \left((A_0 A_1 + B_0 B_1) \sin\left(\frac{s_1}{\epsilon}\right) + \right. \\ &\quad \left. \left. (A_0 B_1 - A_1 B_0) \cos\left(\frac{s_1}{\epsilon}\right) \right)^2 \right], \end{aligned} \quad (61)$$

$$\beta = \frac{e^{2\frac{r(\lambda_1)}{\epsilon}}}{B_0^2} \left[\omega_1^2 \left(\sin\left(\frac{s_1}{\epsilon}\right) \right)^2 + \left(A_1 \sin\left(\frac{s_1}{\epsilon}\right) + B_1 \cos\left(\frac{s_1}{\epsilon}\right) \right)^2 \right]. \quad (62)$$

We remind that $\det(\Psi) = 1$. Computing now $\det(\Psi) = ad - bc$ using the expressions (60) we find

$$\left(\frac{e^{\frac{r(\lambda_1)}{\epsilon}}}{B_0} \right)^2 = \frac{1}{B_0 B_1} \quad (63)$$

and therefore finally, exact to all orders,

$$\begin{aligned} a &= -\frac{1}{\sqrt{B_0 B_1}} \left[A_0 \sin\left(\frac{s_1}{\epsilon}\right) - B_0 \cos\left(\frac{s_1}{\epsilon}\right) \right], \\ b &= \frac{1}{M\sqrt{B_0 B_1}} \sin\left(\frac{s_1}{\epsilon}\right), \\ c &= -\frac{M}{\sqrt{B_0 B_1}} \left[(A_0 A_1 + B_0 B_1) \sin\left(\frac{s_1}{\epsilon}\right) \right. \end{aligned} \quad (64)$$

$$+ (A_0 B_1 - A_1 B_0) \cos\left(\frac{s_1}{\epsilon}\right) \Big],$$

$$d = \frac{1}{\sqrt{B_0 B_1}} \left[A_1 \sin\left(\frac{s_1}{\epsilon}\right) + B_1 \cos\left(\frac{s_1}{\epsilon}\right) \right].$$

Thus we obtain the final result for the expression (9), exact to all orders,

$$\begin{aligned} \alpha + \beta = & \frac{1}{B_0 B_1} \left[\sin^2\left(\frac{s_1}{\epsilon}\right) \left(\frac{B_0^2 B_1^2}{\omega_0^2} + \omega_1^2 \right) + \cos^2\left(\frac{s_1}{\epsilon}\right) \left(B_0^2 \frac{\omega_1^2}{\omega_0^2} + B_1^2 \right) + \right. \\ & \sin^2\left(\frac{s_1}{\epsilon}\right) \left(A_0^2 \frac{\omega_1^2}{\omega_0^2} + \frac{A_0^2 A_1^2}{\omega_0^2} + \frac{2A_0 A_1 B_0 B_1}{\omega_0^2} + A_1^2 \right) + \\ & \cos^2\left(\frac{s_1}{\epsilon}\right) \left(\frac{A_0^2 B_1^2}{\omega_0^2} + \frac{A_1^2 B_0^2}{\omega_0^2} - \frac{2A_0 A_1 B_0 B_1}{\omega_0^2} \right) + \\ & \sin\left(\frac{s_1}{\epsilon}\right) \cos\left(\frac{s_1}{\epsilon}\right) \times \\ & \left. \left(-2A_0 B_0 \frac{\omega_1^2}{\omega_0^2} + 2A_1 B_1 + \frac{2}{\omega_0^2} (A_0 A_1 + B_0 B_1) (A_0 B_1 - A_1 B_0) \right) \right]. \end{aligned} \quad (65)$$

4.2. Leading asymptotic terms in the power expansion in terms of ϵ

So far the result is exact. Let us consider the first order WKB approximation, which is the generic case, that is

$$A(\lambda) \approx \epsilon \sigma'_{1,+}(\lambda), \quad B(\lambda) \approx \frac{\sigma'_{0,+}(\lambda)}{i} = \omega(\lambda). \quad (66)$$

Substituting these values of $A(\lambda)$ and $B(\lambda)$ into (65) we find

$$\begin{aligned} \alpha + \beta = & 2 \frac{\omega_1}{\omega_0} + \\ & \epsilon^2 \left(\frac{\omega_1 \omega'(\lambda_0)^2}{4\omega_0^5} - \frac{\cos\left(\frac{2 \int_{\lambda_0}^{\lambda_1} \omega(x) dx}{\epsilon}\right) \omega'(\lambda_0) \omega'(\lambda_1)}{2\omega_0^3 \omega_1} + \frac{\omega'(\lambda_1)^2}{4\omega_0 \omega_1^3} \right) + O(\epsilon^3). \end{aligned} \quad (67)$$

As we have shown above $\bar{E}_1 = \frac{\alpha+\beta}{2} E_0$ and $\frac{(\Delta E_1)^2}{E_0^2} = \frac{1}{2} \left[\left(\frac{\bar{E}_1}{E_0} \right)^2 - \left(\frac{\omega_1}{\omega_0} \right)^2 \right]$. Therefore

$$\begin{aligned} \frac{(\Delta E_1)^2}{E_0^2} = & \\ & \epsilon^2 \left(\frac{\omega_1^2 \omega'(\lambda_0)^2}{8\omega_0^6} - \frac{\cos\left(\frac{2 \int_{\lambda_0}^{\lambda_1} \omega(x) dx}{\epsilon}\right) \omega'(\lambda_0) \omega'(\lambda_1)}{4\omega_0^4} + \frac{\omega'(\lambda_1)^2}{8\omega_0^2 \omega_1^2} \right) + O(\epsilon^3). \end{aligned} \quad (68)$$

Substituting into the last formula $\omega(\lambda) = \sqrt{1 + \lambda}$ we obtain exactly the approximation (36). Thus the WKB approach yields exactly the leading asymptotic term for \bar{E}_1 and

μ^2 for general $\omega(t)$, and not only that, but also all the higher power terms of ϵ if desired. Below we shall explain how all the higher order WKB terms can be calculated using our closed form formula [7]. Please note that the expression in the big brackets of (68) is positive definite (its minimal value is a complete square of a real quantity), as it must be.

Now we consider the special cases where $\omega'(\lambda_0)$ and $\omega'(\lambda_1)$ vanish which implies that the leading order terms in the above two equations vanish. In such a special case we have to work out the higher order WKB terms. By this reasoning and calculation we prove that if $\omega(t)$ is of class \mathcal{C}^m (all derivatives up to and including the order m are continuous) then $\mu \propto \epsilon^{(m+1)}$.

Therefore we now suppose that all derivatives at λ_0 and λ_1 vanish up to order $(n-1)$

$$\omega'(\lambda_0) = \dots = \omega^{(n-1)}(\lambda_0) = \omega^{(n-1)}(\lambda_1) = 0, \quad \omega^{(n)}(\lambda_0)\omega^{(n)}(\lambda_1) \neq 0. \quad (69)$$

Then

$$\sigma'_1(\lambda_0) = \sigma'_1(\lambda_1) = \dots = \sigma'_{n-1}(\lambda_0) = \sigma'_{n-1}(\lambda_1) = 0, \quad \sigma'_n(\lambda_0)\sigma_n(\lambda_1) \neq 0. \quad (70)$$

There are two cases to be considered separately, namely, the case of odd and the case of even n .

In the case $n = 2k - 1$ in order to find an approximation to μ^2 up to order ϵ^{4k-2} we expand $A(\lambda)$ and $B(\lambda)$ as follows:

$$\begin{aligned} A(\lambda) &= \sum_{u=k}^{2k} \epsilon^{2u-1} \sigma'_{2u-1,+}(\lambda) + O(\epsilon^{4k+1}), \\ B(\lambda) &= \omega(\lambda) - i \sum_{u=k}^{2k-1} \epsilon^{2u} \sigma'_{2u,+}(\lambda) + O(\epsilon^{4k}). \end{aligned} \quad (71)$$

We observe that the terms of the approximation (71) of the order higher than ϵ^{2k} are essential only in the following terms appearing in the brackets of the expression (65):

$$\begin{aligned} &\frac{\cos^2\left(\frac{s_1}{\epsilon}\right) \omega_1^2 B_0^2}{\omega_0^2}, \quad \cos^2\left(\frac{s_1}{\epsilon}\right) B_1^2, \quad \frac{\sin^2\left(\frac{s_1}{\epsilon}\right) B_0^2 B_1^2}{\omega_0^2}, \\ &\sin\left(\frac{s_1}{\epsilon}\right) \cos\left(\frac{s_1}{\epsilon}\right) \left(-2 \frac{\omega_1^2 A_0 B_0}{\omega_0^2} + 2 A_1 B_1 - 2 \frac{A_1 B_1 B_0^2}{\omega_0^2} + 2 \frac{A_0 B_0 B_1^2}{\omega_0^2} \right). \end{aligned} \quad (72)$$

Taking into account that

$$B(\lambda)^2 = \omega(\lambda)^2 - 2\omega(\lambda)i \sum_{u=k}^{2k-1} \epsilon^{2u} \sigma'_{2u,+}(\lambda) + O(\epsilon^{4k}), \quad (73)$$

$$B_0^2 B_1^2 = \omega_0^2 B_1^2 + \omega_1^2 B_0^2 - \omega_0^2 \omega_1^2 + O(\epsilon^{4k}), \quad (74)$$

and

$$A(\lambda)B(\lambda) = \omega(\lambda)A(\lambda) + O(\epsilon^{4k-1}) \quad (75)$$

we easily find:

$$-2\frac{\omega_1^2 A_0 B_0}{\omega_0^2} + 2\frac{A_0 B_0 B_1^2}{\omega_0^2} = O(\epsilon^{4k-1}), \quad (76)$$

$$2A_1 B_1 - 2\frac{A_1 B_1 B_0^2}{\omega_0^2} = O(\epsilon^{4k-1}), \quad (77)$$

yielding that the coefficient of $\sin\left(\frac{s_1}{\epsilon}\right)\cos\left(\frac{s_1}{\epsilon}\right)$ in (72) is of the order $O(\epsilon^{4k-1})$, and we find

$$\begin{aligned} & \frac{\cos^2\left(\frac{s_1}{\epsilon}\right)\omega_1^2 B_0^2}{\omega_0^2} + \cos^2\left(\frac{s_1}{\epsilon}\right)B_1^2 + \frac{\sin^2\left(\frac{s_1}{\epsilon}\right)B_0^2 B_1^2}{\omega_0^2} = \\ & 2\omega_1^2 \cos^2\left(\frac{s_1}{\epsilon}\right) + \omega_1^2 \sin^2\left(\frac{s_1}{\epsilon}\right) + \\ & \sum_{u=k}^{2k-1} \epsilon^{2u} \left(-2\omega_1 i \sigma'_{2u,+}(\lambda_1) - 2i \frac{\omega_1^2}{\omega_0} \sigma'_{2u,+}(\lambda_0) \right) + O(\epsilon^{4k-1}). \end{aligned} \quad (78)$$

Therefore the expression in the brackets of (65) is equal to

$$\begin{aligned} & 2\omega_1^2 + \sum_{u=k}^{2k-1} \epsilon^{2u} \left(-2\omega_1 i \sigma'_{2u,+}(\lambda_1) - 2i \frac{\omega_1^2}{\omega_0} \sigma'_{2u,+}(\lambda_0) \right) + \\ & \epsilon^{4k-2} \left(\frac{\omega_1^2 \sigma'_{2k-1,+}(\lambda_0)^2}{\omega_0^2} + \sigma'_{2k-1,+}(\lambda_0)^2 - \right. \\ & \left. \frac{2\omega_1 \sigma'_{2k-1,+}(\lambda_0) \sigma'_{2k-1,+}(\lambda_1)}{\omega_0} \left(1 - 2 \sin^2\left(\frac{s_1}{\epsilon}\right) \right) \right) + O(\epsilon^{4k-1}). \end{aligned} \quad (79)$$

Noting that

$$\frac{1}{B_0 B_1} = \frac{1}{\omega_0 \omega_1} \left(1 + i \sum_{u=k}^{2k-1} \epsilon^{2u} \left(\frac{\sigma'_{2u,+}(\lambda_0)}{\omega_0} + \frac{\sigma'_{2u,+}(\lambda_1)}{\omega_1} \right) \right) + O(\epsilon^{4k}) \quad (80)$$

we finally obtain

$$\begin{aligned} & \frac{\overline{(\Delta E_1)^2}}{E_0^2} = \epsilon^{4k-2} \left(\frac{\sigma'_{2k-1,+}(\lambda_1)^2}{2\omega_0^2} + \frac{\omega_1^2 \sigma'_{2k-1,+}(\lambda_0)^2}{2\omega_0^4} - \right. \\ & \left. \frac{\omega_1 \sigma'_{2k-1,+}(\lambda_0) \sigma'_{2k-1,+}(\lambda_1)}{\omega_0^3} \cos\left(\frac{2s_1}{\epsilon}\right) \right) + O(\epsilon^{4k-1}). \end{aligned} \quad (81)$$

The second case is when n is even, $n = 2k$, k positive integer, so that we assume now

$$A(\lambda) = \sum_{u=k}^{2k} \epsilon^{2u+1} \sigma'_{2u+1,+}(\lambda) + h.o.t., \quad (82)$$

$$B(\lambda) = \omega(\lambda) - i \sum_{u=k}^{2k} \epsilon^{2u} \sigma'_{2u,+}(\lambda) + h.o.t.$$

Then, similarly as above, using the equalities

$$B(\lambda)^2 = \omega(\lambda)^2 - 2\omega(\lambda)i \sum_{u=k}^{2k-1} \epsilon^{2u} \sigma'_{2u,+}(\lambda) - \epsilon^{4k} (\sigma'_{2k,+}(\lambda))^2 + O(\epsilon^{4k+2}), \quad (83)$$

$$B_0^2 B_1^2 = \omega_0^2 B_1^2 + \omega_1^2 B_0^2 - \omega_0^2 \omega_1^2 - 4\omega_0 \omega_1 \epsilon^{4k} \sigma'_{2k,+}(\lambda_0) \sigma'_{2k,+}(\lambda_1) + O(\epsilon^{4k+2}), \quad (84)$$

$$A(\lambda)B(\lambda) = \omega(\lambda)A(\lambda) + O(\epsilon^{4k+1}) \quad (85)$$

and

$$\begin{aligned} \frac{1}{B_0 B_1} &= \frac{1}{\omega_0 \omega_1} \left(1 + i \sum_{u=k}^{2k} \epsilon^{2u} \left(\frac{\sigma'_{2u,+}(\lambda_0)}{\omega_0} + \frac{\sigma'_{2u,+}(\lambda_1)}{\omega_1} \right) \right) - \\ &\epsilon^{4k} \left(\frac{(\sigma'_{2k,+}(\lambda_0))^2}{\omega_0^2} + \frac{(\sigma'_{2k,+}(\lambda_0))(\sigma'_{2k,+}(\lambda_1))}{\omega_0 \omega_1} + \frac{(\sigma'_{2k,+}(\lambda_1))^2}{\omega_1^2} \right) + O(\epsilon^{4k+1}) \end{aligned} \quad (86)$$

we obtain

$$\begin{aligned} \frac{(\overline{\Delta E_1})^2}{E_0^2} &= -\epsilon^{4k} \left(\frac{(\sigma'_{2k,+}(\lambda_1))^2}{2\omega_0^2} + \frac{\omega_1^2 \sigma'_{2k,+}(\lambda_0)^2}{2\omega_0^4} - \right. \\ &\left. \frac{\omega_1 \sigma'_{2k,+}(\lambda_0) \sigma'_{2k,+}(\lambda_1)}{\omega_0^3} \cos \left(\frac{2s_1}{\epsilon} \right) \right) + O(\epsilon^{4k+1}). \end{aligned} \quad (87)$$

We now observe that the expression

$$\begin{aligned} \frac{(\overline{\Delta E_1})^2}{E_0^2} &= (-1)^{n+1} \epsilon^{2n} \left(\frac{(\sigma'_{n,+}(\lambda_1))^2}{2\omega_0^2} + \frac{\omega_1^2 \sigma'_{n,+}(\lambda_0)^2}{2\omega_0^4} - \right. \\ &\left. \frac{\omega_1 \sigma'_{n,+}(\lambda_0) \sigma'_{n,+}(\lambda_1)}{\omega_0^3} \cos \left(\frac{2s_1}{\epsilon} \right) \right) + O(\epsilon^{2n+1}). \end{aligned} \quad (88)$$

coincides with (81) when $n = 2k - 1$ and with (87) when $n = 2k$. If n -th derivative of $\omega(t)$ at λ_0 and λ_1 is nonzero, it means that $\omega(t)$ is of class \mathcal{C}^m where $m = n - 1$, or $n = m + 1$. Then μ^2 is oscillating as $\epsilon \rightarrow 0$ but in the mean goes to zero as a power $\mu^2 \propto \epsilon^{2(m+1)}$, and therefore the general equation (88) proves our assertion.

4.3. Further simplifications of the general formula for the leading asymptotic term (88)

Now we want to simplify the expression (88) by expressing σ'_n in terms of ω'_n , using our explicit results in [7].

Let $M = \cup_{k=1}^{\infty} \mathbf{N}^k$, \mathbf{N} is the set of non-negative integers. We define the map $L : M \rightarrow \mathbf{N}$ by

$$L(\nu) = 1 \cdot \nu_1 + 2 \cdot \nu_2 + \dots + l \cdot \nu_l \quad (89)$$

and denote by $L(\nu) = m$ the equation

$$L(\nu) = 1 \cdot \nu_1 + 2 \cdot \nu_2 + \dots + m \cdot \nu_m = m, \quad (90)$$

with $m \in \mathbf{N}$, $\nu \in M$ (there is one-to-one correspondence between the set $\{(\nu_1, \dots, \nu_m)\}$ of solutions of (90) and the set of *partitions* of m). For a vector $\nu = (\nu_1, \dots, \nu_l) \in M$ we denote $Q^{(\nu)} = (Q')^{\nu_1} (Q'')^{\nu_2} \dots (Q^{(l)})^{\nu_l}$, $|\nu| = \nu_1 + \dots + \nu_l$ and let $\nu(i)$ ($i = 1, \dots, l-1$) be the vector $(\nu_1, \dots, \nu_i + 1, \nu_{i+1} - 1, \dots, \nu_l)$. It is shown in [7] that the functions σ'_m are of the form

$$\sigma'_m = \sum_{\nu: L(\nu)=m} \frac{U_\nu Q^{m-|\nu|} Q^{(\nu)}}{Q^{\frac{3m-1}{2}}}, \quad (91)$$

where the coefficients U_ν satisfy the recurrence relation

$$U_\nu = \frac{1}{2} \sum_{\mu, \theta \neq 0, \mu + \theta = \nu} U_\mu U_\theta + \frac{(4 - L(\nu) - 2|\nu|) U_{(\nu_1-1, \nu_2, \dots, \nu_l)}}{4} + \sum_{i=1}^{l-1} \frac{(\nu_i + 1) U_{\nu(i)}}{2}, \quad (92)$$

with $U_{\bar{0}} = -1$ and we put $U_\alpha = 0$ if among the coordinates of the vector α there is a negative one.

In our case $Q(x) = -\omega^2(x)$. Therefore in the case when

$$\omega'(\lambda) = \dots = \omega^{(n-1)}(\lambda) = 0, \quad \omega^{(n)}(\lambda) \neq 0 \quad (93)$$

from (91) we obtain

$$\sigma'_n = \frac{U_{\tilde{\nu}} Q^{(n)}}{Q^{\frac{n+1}{2}}}, \quad (94)$$

where $U_{\tilde{\nu}}$ is computed by (92) and $\tilde{\nu} = (0, \dots, 0, 1)$ is the solution of $L(\nu) = n$ with all entries except of the last one equal to zero. From (92) we obtain $U_{\tilde{\nu}} = \frac{1}{2^{n-1}} U_{(1, 0, \dots, 0)} = \frac{1}{2^{n-1}} \left(\frac{1}{4} + \frac{1}{2} U_{\bar{0}} \right) = -\frac{1}{2^{n+1}}$ yieding

$$\sigma'_n = -\frac{Q^{(n)}}{2^{n+1} Q^{\frac{n+1}{2}}}. \quad (95)$$

If (93) holds then

$$Q^{(n)} = -2\omega\omega^{(n)}. \quad (96)$$

Therefore, from (95) we obtain

$$\sigma'_{n,+} = \frac{i^{n+1}\omega^{(n)}}{2^n\omega^n}. \quad (97)$$

Substituting this expression in (88) we obtain the final general formula for the leading asymptotic term of μ^2 , namely

$$\begin{aligned} \frac{(\overline{\Delta E_1})^2}{E_0^2} = & \quad (98) \\ \frac{\epsilon^{2n}}{2^{2n+1}} & \left(\frac{\omega_1^2(\omega_0^{(n)})^2}{\omega_0^{2(n+2)}} + \frac{(\omega_1^{(n)})^2}{(\omega_1)^{2n}\omega_0^2} - 2\frac{\omega_0^{(n)}\omega_1^{(n)}}{\omega_0^{n+3}\omega_1^{n-1}} \cos\left(\frac{2s_1}{\epsilon}\right) \right) + O(\epsilon^{2n+1}). \end{aligned}$$

In the special case $n = 1$ we recover the formula (68). We emphasize that the above expression (98) is indeed evidently positive definite, which must always be the case for the variance.

If the lowest nonvanishing derivative is the n -th derivative of $\omega(t)$ at λ_0 and/or λ_1 , then it means that $\omega(t)$ is of class \mathcal{C}^m where $m = n - 1$, or $n = m + 1$. Then μ^2 is oscillating as $\epsilon \rightarrow 0$ but in the mean goes to zero as a power $\mu^2 \propto \epsilon^{2(m+1)}$, and therefore the general equation (98) proves our assertion.

If $\omega(t)$ is an analytic function on the real time axis $(-\infty, +\infty)$, the decay to zero is oscillating and on the average is exponential $\propto \exp(-\text{const}/\epsilon)$ [3], [17], [18], [19]. This exponential smallness stems from the divergence of the relevant series and has been extensively studied in related works [23], [24], [25], [26] and [27], where the resummation techniques have been devised.

Let us now summarize our results for the variance μ^2 as a function of $\omega(t)$ embodied in the exact general formulae (64) and (65). If $\omega(t)$ is analytic between t_0 and $t_1 \geq t_0$ then the equation (98) applies, and as we see μ^2 is dominated by the switch-on and switch-off events at t_0 and t_1 , respectively. However, the smaller the lowest nonvanishing derivatives of $\omega(t)$ at the two points, the smaller will be the power law contribution. Indeed, if t_0 and t_1 go to $-\infty$ and $+\infty$, respectively, and $\omega(t)$ is analytic on the entire interval, then the behaviour will become exponential at sufficiently large $\epsilon \geq \epsilon_c$, whilst it will be oscillatory and in the mean a power law at small $\epsilon \leq \epsilon_c$. For example, in figure 5 we observe only exponential regime, because ϵ_c is so small.

If there is any non-analyticity of $\omega(t)$ on this interval, say at t_i , then the calculation of μ^2 must be split into two intervals (t_0, t_i) and (t_i, t_1) , calculating the transition matrices for each interval separately, then looking at their product and calculating its matrix elements a, b, c, d , and then proceeding to calculate $\alpha + \beta$ and finally μ^2 , which will obey a power law like in (98).

In other words, if $\omega(t)$ is analytic everywhere, and if the switch-on and switch-off events are completely eliminated, in the sense that they are infinitely smooth, the behaviour of $\mu^2(\epsilon)$ is exponential, as explained above. In all other cases it is a power law.

5. Periodic $\omega(t)$

In case when $\omega(t)$ is periodic with period τ but otherwise completely general we can state some general rigorous results. Since the frequency ω_0 at time t_0 and ω_1 at time $t_1 = t_0 + \tau$ are equal, we see from equation (11) due to positive definiteness of μ^2 that \bar{E}_1 is always greater than E_0 , that is, in a period τ , or any integer multiple of it, $T = n\tau$, the mean energy \bar{E}_1 never decreases.

If we denote by Φ_1 the transition map (4) for one period of our periodic system, then the transition map Φ_n for an interval of exactly n periods of length τ is simply a power of Φ_1 , namely

$$\Phi_n = \Phi_1^n. \quad (99)$$

If we use units such that $\omega_0 = \omega_1 = 1$ and $M = 1$, then the mean energy from (8) and (9) can be expressed elegantly as the trace of the product of the transition matrix Φ and its transpose Φ^T , namely

$$\bar{E}_1 = \frac{E_0}{2}(\alpha + \beta) = \frac{E_0}{2}(a^2 + b^2 + c^2 + d^2) = \frac{E_0}{2}\text{Tr}(\Phi\Phi^T). \quad (100)$$

If $\Phi = \Phi_n = \Phi_1^n$, then we have to understand the behaviour of the above expression (100) as a function of the number of periods n . This is obviously dictated by the eigenvalues of the transition map Φ_1 , whose determinant is of course equal to 1, and its trace is denoted by S , which can be reduced to the diagonal form by a similarity transformation

$$\Phi_1 = WDW^{-1} \quad (101)$$

where W is the transformation matrix and D is the diagonal matrix with the eigenvalues e_1 and $e_2 = 1/e_1$, namely the solutions of the quadratic equation for e ,

$$e^2 - eS + 1 = 0, \quad S = \text{Tr}\Phi_1 \quad e_1 = 1/e_2 = \frac{S}{2} \pm \sqrt{\left(\frac{S}{2}\right)^2 - 1}. \quad (102)$$

Therefore $e_1, e_2 = 1/e_1$ are either real reciprocals (if $|S| > 2$) or complex conjugated numbers on the unit circle (if $|S| < 2$). The n -th power of Φ_1 can then be written as

$$\Phi = \Phi_n = \Phi_1^n = W D^n W^{-1}. \quad (103)$$

Therefore, the matrix elements a, b, c, d of Φ_n are bounded when $|S| < 2$ and oscillate with n , while they increase exponentially when $|S| > 2$. Indeed, if $e_1 > 1$, then the asymptotic behaviour of (100) is

$$\bar{E}_1 \approx K E_0 e_1^{2n}, \quad (104)$$

where K is a constant determined by the matrix W in the transformation (101), and the variance at sufficiently large n goes asymptotically as

$$\mu^2 = \overline{(\Delta E_1)^2} \approx \frac{1}{2} \bar{E}_1^2 \approx \frac{K^2}{2} E_0^2 e_1^{4n}. \quad (105)$$

Therefore, in case when $e_1 > 1$, we indeed find that the mean energy and with it the variance of the energy increase exponentially with time $T = n\tau$, and since $\omega(t)$ is bounded, nothing is conserved. e_1 is determined by $S = \text{Tr}\Phi_1$, and this in turn is determined by the specific properties of the system, that is by the nature of $\omega(t)$ on the interval $(0, \tau)$.

So far we discussed the statistical properties of the energy distribution $P(E_n)$ in a periodic system. Of course, the contour of the initial uniform canonical ensemble \mathcal{K}_0 is topologically always a circle, it evolves into the closed curve \mathcal{K}_n after the n -th full period, with the preserved, constant, area enclosed by \mathcal{K}_n . This curve is just rotating and oscillating with n in case $|\text{Tr}\Phi_1| = |S| < 2$. If $|S| > 2$, the action by $\Phi_n = \Phi_1^n$ is exponentially stretching in the direction of the eigenvector with eigenvalue $e_1 > 1$, and exponentially contracting in the direction of the other eigenvector with the eigenvalue $e_2 = 1/e_1 < 1$. Therefore, the energy of the individual initial condition described by the formula (5) will be exponentially increasing for *any* initial condition (q_0, p_0) (vector in the phase plane (q, p)), except for the case when (q_0, p_0) is exactly in the direction of the second eigenvector (stable manifold) corresponding to $e_2 = 1/e_1 < 1$.

6. General formula for the energy evolution

In this section we wish to consider an exact expression for the evolution of the energy distribution by studying a decomposition of one adiabatic process into several consecutive adiabatic processes.

Let us first mention that the energy distribution $P(E_1)$ evolved from the original delta-like distribution $\delta(E - E_0)$ is a kind of a Green function for the energy evolution. Let us denote it by $G(E_1; E_0)$. If we have a distribution of initial energies $w(E_0)$, such that at each energy E_0 the distribution on the energy contour is a uniform canonical distribution, then the final energy distribution is

$$P(E_1) = \int G(E_1; E_0)w(E_0)dE_0. \quad (106)$$

Thus by knowing G , which we call G -function, we can calculate the final energies of any family $w(E_0)$ of uniform canonical ensembles of initial conditions.

If the adiabatic process is ideal adiabatic, then the G -function is a delta function,

$$G(E_1; E_0) = \delta(E_1 - \omega_1 E_0 / \omega_0). \quad (107)$$

For ensembles of other types, which are not uniform canonical, we must go back to our fundamental equation (7) and perform the averaging using the distribution in space (E_0, ϕ) .

Now suppose that the interval of length T is divided into an arbitrary number of finite subintervals (t_j, t_{j+1}) , where t_0 is the beginning of the process (interval) and t_n is the end of the process, and $j = 0, 1, \dots, n-1$. The behaviour of $\omega(t)$ inside each j -th subinterval is assumed to be entirely arbitrary, but the process must be such that at

each integration step t_j the distribution is uniform canonical. This condition is certainly satisfied if the process is ideal adiabatic, in general not.

It is then obvious that the energy G -function $G(E; E_0)$ for the complete process divided into n subintervals is given by the multiple integral

$$G(E; E_0) = \underbrace{\int \dots \int}_{n-1} G_n(E; x_{n-1}) G_{n-1}(x_{n-1}; x_{n-2}) \dots G_1(x_1; E_0) dx_{n-1} \dots dx_2 dx_1. \quad (108)$$

Indeed, using (107) we can immediately verify this equation. All moments of the final distribution can be easily calculated as they are all fully determined by the first moment alone, so what we need is just the first moment. From our theory in section 2 we know that in full generality the first moment of any $G(E; E_0)$ is a linear function of the initial value E_0 , namely

$$\bar{E} = \int EG(E; E_0) dE = gE_0, \quad (109)$$

where the constant $g = (\alpha + \beta)/2$ is a constant independent of E_0 and is determined by the nature of $\omega(t)$ inside the relevant interval of evolution. We shall call g the g -factor of G . For an ideal adiabatic process we know from (12) that $g_j = \omega_j/\omega_{j-1}$. From equation (108) we immediately find the factorization property

$$g = g_n g_{n-1} \dots g_2 g_1, \quad \bar{E} = gE_0 = g_n \dots g_2 g_1 E_0. \quad (110)$$

Obviously, for an ideal adiabatic process where each $g_j = \omega_j/\omega_{j-1}$, the above equation is certainly satisfied.

It is possible also to show the converse [28]: If the composition formula (108) is true for *any* intermediate points of integration t_j and x_j , then the process must be ideal adiabatic, implying that

$$G(E_j; E_{j-1}) = \delta(E_j - \omega_j E_{j-1} / \omega_{j-1}) \quad (111)$$

applies for all j , and $g_j = \omega_j/\omega_{j-1}$. This can be shown by splitting the time interval (t_0, t_n) into infinitesimal subintervals and using a piecewise constant function to approximate $\omega(t)$, and then using $g_j = \frac{1}{2}(\omega_j^2/\omega_{j-1}^2 + 1)$ from equation (24) for all j , finally evaluating g by equation (110), and finding $g = \omega_n/\omega_0$, which implies that the process is ideal adiabatic at all times of the time interval, because $\mu^2 = 0$.

The composition formula (108) (factorization property of the G -function) will apply also in nonlinear systems, but the relationship between \bar{E}_1 and E_0 is then no longer linear. Therefore using the composition formula (108) for infinitesimal intervals, and approximating $\omega(t)$ by piecewise constant or piecewise linear functions etc. might be of extreme importance to find new global powerful approximations for G -functions and their moments.

The theory for nonlinear systems is left open for the future work.

7. Discussion and conclusions

Our aim in this work is to study the time evolution of the energy in a general (*no restriction upon $\omega(t)$*) time-dependent 1D harmonic oscillator in a rigorous way, and then also to calculate the statistical properties of the final energy distribution $P(E_1)$ after some time of length T , if at the beginning we have a uniform canonical distribution of initial conditions at constant energy E_0 . We are able to calculate rigorously all the moments of $P(E_1)$. Odd moments are exactly zero, the even moments are powers of the second moment and the second moment μ^2 is a function of the first moment. Therefore everything is determined by the first moment \bar{E}_1 and the variance μ^2 , which is a function of \bar{E}_1 . Thus the energy distribution function $P(E_1)$ is universal in the sense that it does not depend on any other properties of $\omega(t)$. (Mathematically speaking, the distribution function $P(E_1)$ is a two parameter family parametrized by \bar{E}_1 and μ^2 , but in physics these two parameters are connected through the formula (11). Of course, the shape of the distribution function $P(E_1)$ does not change if \bar{E}_1 is translationally shifted, and thus for the mathematical purposes we can set $\bar{E}_1 = 0$.) In this analysis we clearly see when the adiabatic invariant $I(t) = E(t)/\omega(t)$ is conserved or not. In the (ideal) adiabatic limit $T \rightarrow \infty$ it is conserved, the variance μ^2 is zero and $E_1 = \bar{E}_1 = \omega_1 E_0 / \omega_0$. If it is not conserved exactly, when T is finite, we find $\mu^2 > 0$, and it can be calculated using a WKB method analytically in a closed form, which is a major achievement of this work. We have also studied three specific solvable models and have demonstrated the power of the WKB expansion, where already the leading WKB term usually very well describes the asymptotic behaviour of μ^2 when $\epsilon = 1/T$ goes to zero. We also show what happens if $\omega(t)$ is smooth and of class \mathcal{C}^m , having m continuous derivatives, calculating and showing that μ^2 oscillates as ϵ goes to zero, but in the mean vanishes as $\propto \epsilon^{2(m+1)}$. If $\omega(t)$ is analytic, thus it also is of class \mathcal{C}^∞ , it is known from the literature that μ^2 must decay exponentially $\propto \exp(\text{const}/\epsilon)$. If $\omega(t)$ is periodic, \bar{E}_1 can grow exponentially, and so does the variance μ^2 , in which case $I(t) = E(t)/\omega(t)$ is not conserved, but we can describe the system. We have introduced the so-called G -function, which is a kind of a Green function for the evolution of the energy and derived a composition formula for it when the interval of evolution is decomposed into a finite number of subintervals, for which the corresponding G_j -function is known for all subintervals j and is uniform canonical there. This formula applies also to nonlinear systems and might be a good starting basis to describe them. The theory for nonlinear systems remains open and is a subject of the current research [28].

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References

- [1] Einstein A 1911 *Inst. intern. phys. Solway, Rapports et discussions* **1** 450
- [2] Robnik M 2005 *Encyclopedia of Nonlinear Science* ed A Scott (New York: Routledge) pp 2–5
- [3] Landau L D and Lifshitz E M 1996 *Mechanics: Course of Theoretical Physics* (Oxford: Butterworth-Heineman)
- [4] Reinhardt W P 1994 *Prog. Theor. Phys. Suppl.* **116** 179
- [5] Henrard J 1993 *Dynamics Reported Vol. 2* Eds. C.K.R.T. Jones, U. Kirchgraber and H.O. Walther (Berlin: Springer) 117-235
- [6] Robnik M and Romanovski V G 2006 *J. Phys. A: Math. Gen.* **39** L35 nlin.CD/0506033
- [7] Robnik M and Romanovski V G 2000 *J. Phys. A: Math. Gen.* **33** 5093
- [8] Kulsrud R M 1957 *Phys. Rev.* **106** 205
- [9] Hertweck F and Schluter A 1957 *Z. Naturforschung* **12A** 844
- [10] Lenard A 1959 *Ann. Phys., NY* **6** 261
- [11] Gardner C S 1959 *Phys. Rev.* **115** 791
- [12] Courant E D and Snyder H S 1958 *Ann. Phys., NY* **3** 1
- [13] Littlewood J E 1963 *Ann. Phys., NY* **21** 233
- [14] Kruskal M 1962 *J. Math. Phys.* **3** 806
- [15] Lewis H R 1968 *J. Math. Phys.* **9** 1976
- [16] Symon K R 1970 *J. Math. Phys.* **11** 1320
- [17] Knorr G and Pfirsch D 1966 *Z. Naturforschung* **21** 688
- [18] Meyer R E 1973 *Z. angew. Math. Phys.* **24** 293
- [19] Meyer R E 1973 *Z. angew. Math. Phys.* **24** 517
- [20] Gradshteyn I S and Ryzhik I M 1994 *Table of Integrals, Series and Products* 5th Edition, Ed. A. Jeffrey (Boston: Academic Press) pp 159-160
- [21] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (New York: Dover) p 823
- [22] Fröman N 1966 *Arkiv för Physik* **32** 541
- [23] Berry M V 1982 *J. Phys. A: Math. Gen.* **15** 3693
- [24] Berry M V 1990 *Proc. Roy. Soc. Lond.* **A429** 61
- [25] Joye A 1993 *J. Phys. A: Math. Gen.* **26** 6517
- [26] Joye A, Kunz H and Pfister C.-E. 1991 *Ann. Phys. N.Y* **208** 299
- [27] Lim R and Berry M V 1991 *J. Phys. A: Math. Gen.* **24** 3255
- [28] Robnik M and Romanovski V G 2006, in preparation